Advancements in `(cr)libm` development

Presentation at Intel - Portland

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Introduction

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Correct rounding of x^y

Automatic implementation of `libm functions`

Conclusion

Correctly rounded elementary functions - crlibm

crlibm[1]: correctly rounded elementary function library

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- Challenge: Correct rounding requires high accuracy and complete proofs

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This talk goes on...

- Advancements in the correct rounding of x^y
- Techniques for automatic implementation of `libm` functions.

Correct rounding of x^y

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- Correct rounding must overcome the Table Maker's Dilemma

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  - roughly 2^{112} valid inputs
  - Worst-case search of $\overline{\varepsilon}$ currently untractable

Correct rounding of x^n

- Consider x^n, $x \in \mathbb{F}$, $n \in \mathbb{N}$, n small
- Lefèvre: traditional worst-case search is possible
  - Consider each n separately
  - Current range achieved: $n \leq 255$
  - Worst case $\overline{\varepsilon} = 2^{-117}$ comparable to other double precision functions
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- This research paves the road for x^y

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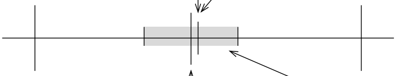
  For ensuring termination, rounding boundary cases must be filtered out



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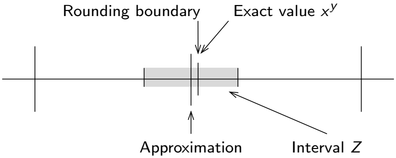

- Rounding boundary cases:
  Complex set for x^y:

$$RB = \{x^y = z | x, y \in \mathbb{F}_{53}, z \in \mathbb{F}_{54}\}$$

Previous approaches

Previous approaches:

- Rewrite

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as

$$x = 2^E \cdot m, \quad y = 2^F \cdot n, \quad z = 2^G \cdot k$$

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- Mainly test whether

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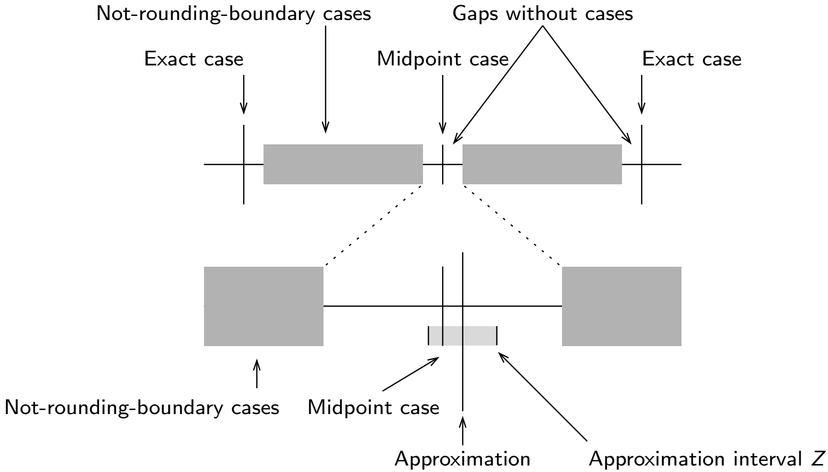
- Cost of the test in double precision:
  - up to 5 square root extractions
  - up to 10 doubled precision multiplies
  - pipeline broken by many `ifs`

An efficient rounding boundary test for $x^y - 1$

Use worst-case information for rounding boundary testing



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  - Still more optimization: 99.1% of RB cases imply $y = \frac{3}{2}$

An efficient rounding boundary test for x^y – 3

Details can be found at

`http://prunel.ccsd.cnrs.fr/ensl-00169409/`

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- duration: a Ph.D. thesis

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Function development by Arénaire members – 2

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- duration: a summer intern-ship at Intel Nizhny Novgorod

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- Maple scripts generating header files
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- Hand-written Gappa proofs
- duration: about 1 month per function

Function development at Intel

And at Intel?

How many man-hours are accounted per `libm` function?

What is the issue?

Why is the Arénaire development process so slow?

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Why is the Arénaire development process so slow?

Actually, I thought we were always doing the same things...

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  - Ch. Lauter (implementation and proof part)
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  - `C`, `C++`
  - Shell scripts
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  - `C`, `C++`
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  - an internal language: `arenaireplot`
- Targetted to
  - portable `C` implementations
  - using double, double-double and triple-double arithmetic
  - with easy-to-handle Horner evaluation

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Let' try it out...

Results on new functions

Last functions in `crlibm`

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Could this be interesting for Intel's customers?

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Conclusion

Introduction

Correct rounding of x^y

Automatic implementation of `libm functions`

Conclusion

Conclusion and outlooks...

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Thank you!

Thank you for your attention !

Questions ?