CS3350 Automata, Computability and Formal Languages
Spring 2026. Homework Assignment 3
Due: 04/30/2026 11:59PM MDT
Individual assignment

Provide answers, explanations and proofs for the following questions in the form of a write-up, which, for technical reasons, can be hand-written and scanned.

Submit your write-up in the form of physical paper to your instructor or as a PDF by email to

`utep-spring-2026-automata-hw3@christoph-lauter.org`

If you submit by email, you must submit in the form of a PDF. No other document formats are accepted. Please ensure as well that your PDF document does not exceed 5MB.

1. Define a 3-tape Turing machine A that does the following:
   - On its first and second tape, A receives two strings w and v, $w, v \in \{0,1\}^\star$, representing two integer numbers. When the machine A is started, the tape heads are located on the left-most position, on the most significant bits of w and v.
   - If none of the inputs w and v is the empty word, the Turing machine A writes the binary representation of the sum of the two integer numbers w and v on the third tape and then accepts the input. If one of w or v is the empty word, the Turing machine does not write anything on the third tape and rejects the input. If the machine A outputs something, it outputs the binary representation of the sum in such a way that the left-most bit contains the most significant bit of the sum. *Be careful: w and v may be of different length.*
2. Run your Turing machine A defined on the example input of 42 (i.e. $w = 101010$) and 7 (i.e. $v = 111$). Give a table with four columns, the first colum containing the state the machine A is in at each step, and the three other columns representing the contents of the tapes and the positions of the tape heads (draw an arrow).
3. Define a 3-tape Turing machine M that implements multiplication, in a manner similar to how A implements addition.
4. Run your Turing machine M defined on the example input of 3 (i.e. $w = 11$) and 8 (i.e. $v = 1000$). Give a table with four columns, again.
5. Write an argument convincing your instructor that you could come up with a Turing machines S for subtraction and a Turing machine D for (Euclidian/integer) division if you were given enough time and you were brave enough to actually describe the δ transition functions needed each time. You may use pseudo-code implementing subtraction and division and refer to the Church-Turing-Thesis to sound extra convincing.
6. Write an argument, backed with pseudo-code, that convinces your instructor that you could define 6-tape Turing machines that read two rational numbers a/b and c/d given in binary on their first 4 tapes (a is on the first tape, b is on the second tape, c on the third, d on the fourth) and that would write the sum $\frac{ad+cb}{bd}$ resp. the product $\frac{ac}{bd}$ on the fifth (numerator) and sixth (denominator) tape of the machine, still in binary.
7. Prove that the set of the real numbers $\mathbb{R}$ is not countable. Using some Gödel numbering argument, deduce a corollary stating that there are "more" real numbers than Turing machines.
8. Show, with a convincing argument that you base on Bellard's formula below, that it is nevertheless possible to define a 3-tape Turing machine that reads an integer n (written in binary) from its first tape and writes a rational approximation a_n/b_n of the real (and transcendental) number π on the second and third tape such that $|a_n/b_n - \pi| \leq c^{-n}$ for all $n \geq 1$ and some fixed $c > 1$.

   Bellard's formula is:

$$\pi = \lim_{n\to\infty} \frac{1}{2^6} \sum_{i=0}^{n} \frac{(-1)^i}{2^{10i}} \left(\frac{-2^5}{4i+1} - \frac{1}{4i+3} + \frac{2^8}{10i+1} - \frac{2^6}{10i+3} - \frac{2^2}{10i+5} - \frac{2^2}{10i+7} + \frac{1}{10i+9} \right)$$